16.5 - Curl and Divergence

In this section we give two new uses of the symbol ∇. First, let's properly define it:

Def: The operator ∇ (pronounced "del") is given by

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$\left(\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \text{ in } \mathbb{R}^2 \right)$.

We are already familiar with one use of ∇:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle, \quad f = f(x,y,z).$$

(Sometimes ∇f is written grad f.) Since ∇ looks like a vector, perhaps we can use it on other types of functions, like vector fields. Between vectors, we can take dot products & cross products (for vectors in $\mathbb{R}^3$), so:

Def: Given a vector field $\vec{F}(x,y,z) = \langle P, Q, R \rangle$, we define:

- the divergence of $\vec{F}$:

$$\boxed{\operatorname{div} \vec{F} = \nabla \cdot \vec{F}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

- the curl of $\vec{F}$:

$$\boxed{\operatorname{curl} \vec{F} = \nabla \times \vec{F}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

(It is worth pointing out that for a vector field $\vec{F}(x,y)$ on $\mathbb{R}^2$, div $\vec{F}$ can be defined, but curl $\vec{F}$ CANNOT!)

Let's do an example before saying more:

Ex: Find the curl and divergence of

$$\vec{F}(x,y,z) = \langle xy^2z^3, x^3yz^2, x^2y^3z \rangle$$

Sol:

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^3 & x^3yz^2 & x^2y^3z \end{vmatrix}$$

$$= \langle 3x^2y^2z - 2x^3yz,\ 3xy^2z^2 - 2xy^3z,\ 3x^2yz^2 - 2xyz^3 \rangle$$

$$= xyz\langle 3xy - 2x^2,\ 3yz - 2y^2,\ 3xz - 2z^2 \rangle$$

$$\text{div}\,\vec{F} = y^2z^3 + x^3z^2 + x^2y^3$$

$\diamond$

What exactly are div $\vec{F}$ and curl $\vec{F}$?

Suppose $\vec{F}$ represents the velocity field of a flowing fluid. The divergence at a point P represents the net rate of change of the mass of fluid flowing from P per unit volume, i.e., div $\vec{F}(P)$ measures the tendency of the fluid to diverge from P. If div $\vec{F} = 0$, we say $\vec{F}$ is incompressible.

The curl of $\vec{F}$ at P, as its name suggests, measures whether particles rotate about an axis at that point, e.g., an eddy or a whirlpool. If they do rotate about an axis at P, then curl $\vec{F}(P)$ gives the direction of the axis, and $|\text{curl } \vec{F}(P)|$ gives a measure of the speed at which they rotate. If curl $\vec{F} = \vec{0}$ at P, then we say $\vec{F}$ is <u>irrotational</u> at P.

Now, div, grad, and curl have some nice relations:

<u>Thm</u>: Let $\vec{F} = \vec{F}(x,y,z)$ & $f = f(x,y,z)$ and suppose that f and the component functions of $\vec{F}$ are C^2. Then:

i) $\text{curl}(\text{grad } f) = \text{curl}(\nabla f) = \nabla \times (\nabla f) = \vec{0}$

ii) $\text{div}(\text{curl } \vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = \vec{0}$

<u>proof</u>:

$$\text{curl}(\nabla f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \langle f_{yz} - f_{zy}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle \underset{\uparrow}{=} \vec{0}$$

by Clairaut's theorem.

$$\operatorname{curl} \vec{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

$$\operatorname{div}(\operatorname{curl} \vec{F}) = \left(\cancel{\frac{\partial^2 R}{\partial x \partial y}} - \cancel{\frac{\partial^2 Q}{\partial x \partial z}}\right) + \left(\cancel{\frac{\partial^2 P}{\partial y \partial z}} - \cancel{\frac{\partial^2 R}{\partial y \partial x}}\right) + \left(\cancel{\frac{\partial^2 Q}{\partial z \partial x}} - \cancel{\frac{\partial^2 P}{\partial z \partial y}}\right)$$

$\square$

A useful consequence of part (i) is:

<u>Thm</u>: If $\vec{F} = \langle P, Q, R \rangle$ is defined on a simply connected region, and if P, Q, R are C^1 on this region, if $\operatorname{curl} \vec{F} = \vec{0}$, then $\vec{F}$ is conservative.

<u>Application of div and curl:</u>

Suppose we are in some region Ω with an electric field $\vec{E}$, magnetic field $\vec{B}$, charge density ρ, and electric current density $\vec{J}$, then

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}, \quad \nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \nabla \times \vec{B} = \mu_0\left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

are known as <u>Maxwell's equations.</u>

Vector Formulations of Green's Theorem

Here we consider $\vec{F}(x,y,z) = \langle P(x,y), Q(x,y), 0 \rangle$.

Then,

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left\langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle,$$

so $(\operatorname{curl} \vec{F}) \cdot \hat{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. Then the equation in Green's theorem reads:

$$\oint_C P\,dx + Q\,dy = \oint_C \vec{F} \cdot d\vec{r} = \iint_D (\operatorname{curl} \vec{F}) \cdot \hat{k}\, dA$$

Recall that $\oint_C \vec{F} \cdot d\vec{r} = \oint_C (\vec{F} \cdot \vec{T})\, ds$. This means that we are integrating the tangential component of $\vec{F}$ along C. What if we wish to integrate the normal component of $\vec{F}$? Recall, if C is parametrized by, $\vec{r}(t) = \langle x(t), y(t) \rangle, a \le t \le b$, then $\vec{T}(t) = \frac{1}{|\vec{r}'(t)|} \langle x'(t), y'(t), 0 \rangle$. You can check that the outward unit normal to C is $\vec{n}(t) = \frac{1}{|\vec{r}'(t)|} \langle y'(t), -x'(t), 0 \rangle$.

Then,

$$\oint_C (\vec{F}\cdot\vec{n})\,ds = \int_a^b (\vec{F}\cdot\vec{n})(t)\,|\vec{r}\,'(t)|\,dt$$

$$= \int_a^b \langle P(\vec{r}(t)), Q(\vec{r}(t)), 0\rangle \cdot \langle y'(t), -x'(t), 0\rangle\,dt$$

$$= \int_a^b \left(P(\vec{r}(t))\,y'(t)\,dt - Q(\vec{r}(t))\,x'(t)\,dt\right)$$

$$= \oint_C P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA$$

$$= \iint_D \left(\operatorname{div} \vec{F}\right) dA$$